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Spectral Lattices

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Abstract Spectral orthomorphisms between the spectral lattices of JBW algebras which preserve the scales extend to Jordan homomorphisms for a large class of algebras. Spectral lattice homomorphism is automatically a σ-lattice homomorphism. The range projection map is, up to a Jordan homomorphism, the only natural map from the spectral lattice onto the projection lattice. Continuity of the range projection determines finiteness of the algebra in Murray–von Neumann comparison theory.

Keywords Spectral order · Spectral orthomorphism · Spectral and projection lattices · Range projection map

1 Introduction and Preliminaries

The aim of this paper is to summarize and deepen recent results on the various lattice theoretic aspects of the positive part of the unit ball of von Neumann algebras and Jordan algebras.

Let $B(H)$ be the algebra of all bounded operators acting on a given Hilbert space H. For self-adjoint operators $x, y \in B(H)$, we write $x \leq y$ if

$$(x\xi, \xi) \leq (y\xi, \xi) \quad \text{for all } \xi \in H.$$

Endowed with this standard order, $\leq$, the self-adjoint part $B(H)_{sa}$ of $B(H)$ becomes an ordered vector space. According to classical result of Kadison in [9] $(B(H)_{sa}, \leq)$ is a poset which is far from being a lattice. More precisely, if H has dimension at least 2, then the supremum $x \vee y$ exists in $(B(H)_{sa}, \leq)$ if, and only if, $x \leq y$ or $y \leq x$. A related result of Sherman states that the self-adjoint part of a C^*-algebra A is a lattice with respect to standard operator order if, and only if, A is Abelian (see [13]). The central topic of this

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paper is another order, $\leq_S$, which may be naturally introduced on the operators. It is called the spectral order and was introduced by Olson in [12]. For a given $x \in B(H)_{sa}$, let E^x_λ denote the spectral projection of x corresponding to the interval $(-\infty, \lambda]$ $(\lambda \in \mathbb{R})$. We define $x \leq_S y$ for $x, y \in B(H)_{sa}$ if

$$E^x_\lambda \geq E^y_\lambda \quad \text{for each } \lambda \in \mathbb{R}.$$

A nice characterization of the spectral order in terms of the moments is due to Olson: For positive operators x and y we have $x \leq_s y$ if, and only if, $x^n \leq y^n$ for each integer n (see [12]). It implies that the spectral order is stronger than the standard order. On the other hand, both orders coincide on mutually commuting elements or provided that one of the elements is a projection. The spectral order has a natural physical interpretation. Let $x \leq_S y$ for quantum observables x and y. Suppose that the state of the system is φ. Then $f(\lambda) = \varphi(E^x_\lambda)$ and $g(\lambda) = \varphi(E^y_\lambda)$ is a distribution function of x and y, respectively. Now the relation $x \leq_S y$ reads as the pointwise ordering $f(\lambda) \geq g(\lambda)$ $(\lambda \in \mathbb{R})$ for all distribution functions which may be associated with the observables x and y.

Let us remark that different type of nonclassical operator order is studied in [5].

Throughout the paper, let M be a JBW algebra with the product $\circ$ (for all details on JBW algebras we refer the reader to [8]). Put

$$E(M) = \{x \in M \mid 0 \leq x \leq 1\}.$$

In contrast to the standard order, the spectral order has the advantage of organizing $E(M)$ into a complete lattice. In fact, the supremum,

$$z = \bigsqcup_{y \in S} y$$

of a set $S \subset E(M)$ in the spectral order is given by the spectral resolution

$$E^z_\lambda = \bigwedge_{x \in S} E^x_\lambda, \quad \lambda \in \mathbb{R}.$$

We shall refer to the structure $(E(M), \leq_S)$ as to the *spectral lattice*. The spectral lattice $E(M)$ contains the projection lattice $P(M) = \{p \in M \mid p = p^2\}$ as a complete sublattice. From this point of view the spectral lattice seems to be a natural "unsharp" extension of the projection lattice. In the present paper we shall study the interplay between the following structures: $(E(M), \leq_S)$ (spectral lattice), $(P(M) \leq)$ (projection lattice), and $(E(M), \leq)$ (effect algebra). In the next section we shall present the results on orthomorphisms of the projection lattices. In particular we show that some important orthomorphisms between the projection lattices are given by the Jordan homomorphisms and that algebras with isomorphic spectral lattices must have isomorphic projection lattices as well.

A natural map connecting the spectral lattice and the projection lattice is the range projection map $x \mapsto r(x)$ assigning to each positive contraction x in M its range projection $r(x)$. If the algebra is commutative, then it is isomorphic to the algebra $C(X)$ of continuous functions on a compact space X. For $f \in C(X)$, the range projection $r(f)$ corresponds to the support of the function f. Leading by this example we can view range projection as a "localization" of a given quantum observable. In the concluding part of this paper we shall prove a characterization of the range projection map in terms of order properties and show that modularity of the projection lattice is equivalent to the fact that the range projection map is a spectral lattice homomorphism.

2 Symmetries of Spectral Lattices

Let $x \in M$. The range projection, $r(x)$, of x is the smallest projection $p \in M$ such that $p \circ x = x$. For $x \in M$, it can be given by the following limit in the strong operator topology $r(x) = \lim_{n\to\infty} x^{1/n}$. If M happens to act on a Hilbert space H, then $r(x)$ is a projection onto the closed span of the range of the operator x. The range projection enables one to extend the relation of orthogonality from the projection lattices to the positive parts of the unit balls of the algebras. We say that x and y are *orthogonal* (in symbols $x \perp y$) if $r(x) \circ r(y) = 0$.

Definition 2.1 Let M_1 and M_2 be JBW algebras. A map $\varphi : E(M_1) \mapsto E(M_2)$ is called a *spectral orthomorphism* if the following two conditions are satisfied:

(i) $\varphi(x) \leq_S \varphi(y)$ whenever $x \leq_S y$,
(ii) $\varphi(x) \perp \varphi(y)$ and $\varphi(x \sqcup y) = \varphi(x) \sqcup \varphi(y)$ whenever $x \perp y$.

If a spectral orthomorphism φ is a bijection and φ^{-1} is a spectral orthomorphism as well, then φ is called a *spectral orthoisomorphism*.

The orthomorphims between the projection lattices are defined as maps preserving the order, orthogonality of projections, and suprema of orthogonal projections. Wigner's theorem states that every orthoisomorphism of the projection lattice $P(B(H))$, where $\dim H \geq 3$, is implemented either by a unitary or an antiunitary map. A more general result along this line is the Dye's theorem saying that any orthoisomorphism of a von Neumann projection lattice without Type I_2 direct summand extends to a Jordan isomorphism of the self-adjoint part of the algebra (see [4]). (Let us recall that a Jordan homomorphism is a linear map $\pi : M_1 \to M_2$ such that $\pi(x^2) = \pi(x)^2$ for all $x \in M_1$.) Dye's theorem was extended by Bunce and Wright to the following very general form: Any orthomorphism $\varphi : P(M_1) \to P(M_2)$, where M_1 and M_2 are JBW algebras and M_1 does not have Type I_2 part, extends to Jordan homomorphism $\pi : M_1 \mapsto M_2$ (see [1]). In summary, symmetries of the projection lattices have been completely described and correspond to the (linear) symmetries of whole algebras. The situation in case of the spectral lattices is different. First of all, there is a spectral orthoisomorphism $x \to x^2$ which is not linear and therefore does not extend to any linear map. Another example of a nonlinear orthomorphism is the map $E(M) \mapsto E(M)$ which assigns to a given element its range projection. However, we have identified that the only obstacle for the existence of such an extension is the property of not preserving the scales (multiples of identity). The following main result has been proved in [7].

Theorem 2.2 *Let M_1 be a JBW algebra with no Type I_2 direct summand and M_2 an arbitrary JBW algebra. Suppose that $\varphi : E(M_1) \mapsto E(M_2)$ is a spectral orthomorphism satisfying the following condition*:

$$\varphi(\lambda 1_{M1}) = \lambda 1_{M_2} \quad \textit{for each } 0 \leq \lambda \leq 1.$$

Then there is a Jordan homomorphism $\pi : M_1 \mapsto M_2$ which extends π.

This result may be viewed as an extension of Dye's theorem to larger structures which contain projection lattices and also "unsharp elements". As the example above indicates even spectral orthoisomorphism may not be linearizable. Nevertheless, it turns out that if two JBW algebras have isomorphic spectral lattices then they have isomorphic projection lattices. Consequently, the algebras with the same spectral lattice are (linearly) isomorphic on condition that the Generalized Gleason's Theorem applies.

Proposition 2.3 *Let M_1 and M_2 be JBW algebras. If there is a spectral orthoisomorphism mapping $E(M_1)$ onto $E(M_2)$, then the projection lattices $P(M_1)$ and $P(M_2)$ are orthoisomorphic.*

Proof Let τ be an orthoisomorphism mapping $E(M_1)$ onto $E(M_2)$. Then, as the identity is the largest element in the positive part of the unit ball, we see that $\tau(1_{M_1}) = 1_{M_2}$. Let us pick a projection $p \in P(M_1)$. Since the spectral order supremum of orthogonal elements is their sum, we have that

$$1_{M_2} = \tau(1_{M_1}) = \tau(p \sqcup (1-p)) = \tau(p) \sqcup \tau(1-p) = \tau(p) + \tau(1-p).$$

However, as $\tau(p)$ and $\tau(1-p)$ are orthogonal, they lie in the orthogonal hereditary algebras $U_{r(\tau(p))}(M)$ and $U_{r(\tau(1-p))}(M)$, respectively. ($U_x(y) = 2x \circ (x \circ y) - x^2 \circ y$.) Therefore, by multiplying the equation above by $\tau(p)$, we obtain

$$\tau(p) = \tau(p)^2.$$

In other words, $\tau(p)$ is a projection. Hence, τ maps the projection lattice $P(M_1)$ onto $P(M_2)$ and induces an isomorphism between the projection lattices. □

A surprising fact about projection lattice orthomorphism, proved by Bunce and Hamhalter in [2], says that a lattice homomorphism between von Neumann algebra projection lattices is always a σ-lattice homomorphism. Based on this fact a similar statement can be proved for the spectral order (see [7]).

Theorem 2.4 *Let M_1 be a self-adjoint part of a von Neumann algebra not containing Type I_2 direct summand and nonzero Abelian direct summand. Let $\varphi : E(M_1) \mapsto E(M_2)$ be a spectral orthomorphism such that*

$$\varphi(\lambda 1_{M_1}) = \lambda 1_{M_1}.$$

The following conditions are equivalent:

(i) $\varphi(x \sqcup y) = \varphi(x) \sqcup \varphi(y)$ *for all* $x, y \in E(M_1)$.
(ii) $\varphi(p \sqcup q) = \varphi(p) \sqcup \varphi(q)$ *for all* $p, q \in P(M_1)$.
(iii) $\varphi(\bigsqcup_{n=1}^{\infty} x_n) = \bigsqcup_{n=1}^{\infty} \varphi(x_n)$, *whenever* $(x_n) \subset E(M_1)$.
(iv) $\varphi(\bigsqcup_{n=1}^{\infty} p_n) = \bigsqcup_{n=1}^{\infty} \varphi(p_n)$, *whenever* $(p_n) \subset P(M_1)$.

3 Range Projections

The *range projection*, $r(x)$, of x is the smallest projection in $P(M)$ with the property $r(x) \circ x = x$. In other words, $r(x) = 1 - E_0^x$. If $x \in E(M)$, then $r(x)$ is the smallest projection majorizing x. Obviously, the range projection is monotone: $x \leq y$ implies $r(x) \leq r(y)$.

In this section we demonstrate that the lattice theoretic properties of the range projection map are intimately connected with the structure of the ambient algebra, in particular with the modularity of the projection lattice. The following property of the range projection map $a \to r(a)$ constitutes a key part of its characterization: Let $a = \sum_n a_n \in E(M)$ for $(a_n) \subset E(M)$ (the sum is supposed to converge in the norm topology). Then $r(a) = \bigvee_n r(a_n)$. For this, by the monotonicity of the range projection we have that $r(a) \geq \bigvee_n r(a_n)$. Suppose that q is a projection less than $r(a)$ which is orthogonal to each $r(a_n)$. Then q is orthogonal to each a_n and also $q \circ a = 0$. Consequently, $a \in U_{1-q}(M)$ and so $r(a) \in U_{1-q}(M)$, which is equivalent to $r(a) \circ q = 0$. (Recall that $U_x(y) = 2x \circ (x \circ y) - x^2 \circ y$.) Hence, $q = 0$. The following definition is an abstraction of the basic properties of the range projection.

Definition 3.1 A map $h\colon E(M) \to P(M)$ is called a *homogeneous projection-valued measure* on $E(M)$ if the following conditions are satisfied:

(i) $h(\lambda a) = h(a)$ whenever $\lambda \in (0, 1]$ and $a \in E(M)$.
(ii) $h(a_1)$ and $h(a_2)$ are orthogonal projections whenever a_1 and a_2 are orthogonal elements in $E(M)$.
(iii) $h(\sum_{n=1}^{\infty} a_n) = \bigvee_n h(a_n)$, whenever $(a_n) \subset E(M)$ is a sequence of mutually operator commuting elements with $\sum_{n=1}^{\infty} a_n \leq 1$. (The sum is supposed to converge in the norm topology.)

The *quasi-Jordan homomorphism* is a (not necessarily linear) map of a JBW algebra A into a JBW algebra B which is a Jordan homomorphism when restricted to any associative subalgebra of A.

Proposition 3.2 *A map $h : E(M) \mapsto P(M)$ is a homogeneous projection-valued measure if, and only if, there is a quasi-Jordan homomorphism $\pi : M \to M$ such that*

$$h(a) = r(\pi(a)) \quad \textit{for all } a \in E(M).$$

Proof Suppose that h is a homogeneous projection-valued map. The restriction of h to $P(M)$ is a (finitely additive) measure on $P(M)$ with values in $P(M)$. By the standard argument (see e.g. [6]) there is a quasi-Jordan homomorphism π extending $h|P(M)$. Let $x \in E(M)$. Then there is a sequence (p_n) of operator commuting projections such that $x = \sum_{n=1}^{\infty} \frac{1}{2^n} p_n$. Hence, $h(x) = \bigvee_{n=1}^{\infty} \pi(p_n)$. On the other hand, as $\pi(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \pi(p_n)$, we see that $r(\pi(x)) = \bigvee_{n=1}^{\infty} \pi(p_n) = h(x)$. The reverse implication can be verified easily. □

As a corollary we have that any homogeneous map h such that $x \leq h(x)$ for each $x \in E(M)$ must be the range projection map as π in the previous result is identity. A map $\tau : M \to M$ is called σ-additive if $\tau(\sum_n p_n) = \sum_n \tau(p_n)$ for each orthogonal sequence $(p_n) \subset P(M)$. (The sums converge in the strong operator topology.) Further, τ is called monotone if $a \leq b$ implies $\tau(a) \leq \tau(b)$.

Proposition 3.3 *Let $h\colon E(M) \to P(M)$, where M has no Type I_2 direct summand. The following conditions are equivalent*:

(i) *h is a monotone σ-additive map which satisfies $h(\lambda x) = h(x)$ for all $x \in E(M)$ and $\lambda \in (0, 1]$.*
(ii) *There is a σ-additive quasi-Jordan homomorphism π of M such that*

$$h(x) = r(\pi(x)) \quad \textit{for all } x \in E(M).$$

Proof (i) $\Rightarrow$ (ii) Suppose that (i) holds. Since $r(x) \geq x \geq \frac{1}{n} E^x_{(1/n,1]}$ for all $x \in E(M)$, we obtain from (i)

$$h(r(x)) \geq h(x) \geq h(E^x_{(1/n,1]}). \tag{1}$$

By the virtue of the σ-additivity of h on the projection lattice, the passage $n \to \infty$ in (1) gives $h(x) = h(r(x))$. Therefore, if π is a quasi-Jordan homomorphism of M extending $h|P(M)$, we have $h(x) = \pi(r(x))$ for all $x \in E(M)$. As π is σ-additive, it is normal when restricted to any σ-finite subalgebra. Especially, it is normal on any singly generated algebra $JBW(x)$, $x \in M$. Hence, in the strong operator topology,

$$\pi(r(x)) = \lim_n \pi(x^{1/n}) = \lim_n \pi(x)^{1/n} = r(\pi(x)).$$

The implication (ii) $\Rightarrow$ (i) is obvious. □

By the result of Bunce and Wright in [1] every quasi-Jordan homomorphism $\pi : M_1 \mapsto M_2$ is linear provided that the algebra M_1 has no Type I_2 direct summand. In this case the previous propositions improve considerably. Using automatic σ-continuity proved in [2], we have the following description of σ-additive homogeneous projection valued maps.

Theorem 3.4 *Let M be the self-adjoint part of a von Neumann algebra $\mathcal{M}$ not containing Type I_2 direct summand and without nonzero Abelian direct summand. Let $h : E(M) \to P(M)$ be a homogeneous projection-valued measure such that*

$$h(e \vee f) = h(e) \vee h(f) \quad \textit{for all } e, f \in P(M).$$

Then

$$h(x) = r(\pi(x)) \quad \textit{for all } x \in E(M),$$

where $\pi = \pi_1 \oplus \pi_2$ is a direct sum of a σ-additive $*$-homomorphism π_1 and a σ-additive $*$-antihomomorphism π_2 of $\mathcal{M}$.

Proof As h restricts to a measure of $P(M)$ into $P(M)$ which preserves the suprema of the projections, we infer by [2] that $h|P(M)$ extends to a σ-additive Jordan homomorphism π of M. The rest follows from Proposition 3.3 and the representation of a $*$-Jordan homomorphisms of a von Neumann algebra by a sum of $*$-homomorphisms and $*$-antihomomorphisms due to Kadison (see [10]). □

The following result, obtained in [3] and [7], shows that the projection lattice is modular if, and only if, the range projection map preserves infima in $E(M)$ with respect to standard (or spectral) order. This extends the existing list of various characterizations of finite von Neumann algebras. Kaplansky theorem proved in [11] which says that a complete orthocomplemented modular lattice is a continuous geometry is one of the important ingredients of the proof.

Theorem 3.5 *Let M be a JBW algebra. The following conditions are equivalent*:

(i) *M is modular.*
(ii) *The range projection preserves infima of elements in $(E(M), \leq)$.*
(iii) *The range projection is a (complete) lattice homomorphism of $(E(M), \leq_S)$ onto $P(M)$.*

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References

1. Bunce, L.J., Wright, J.D.M.: On Dye's theorem for Jordan operator algebras. Expo. Math. **11**, 91–95 (1993)
2. Bunce, L.J., Hamhalter, J.: Countably additive homomorphisms between von Neumann algebras. Proc. Am. Math. Soc. **123**, 3437–3441 (1995)
3. Cattaneo, G., Hamhalter, J.: De Morgan property for effect algebras of von Neumann algebras. Lett. Math. Phys. **59**, 243–252 (2002)
4. Dye, H.A.: On the geometry of projections in certain operator algebras. Ann. Math. **61**, 73–89 (1955)
5. Gudder, S.: An order for quantum observables. Math. Slovaca **56**, 573–589 (2006)
6. Hamhalter, J.: Quantum Measure Theory. Kluwer Academic, Dordrecht (2003)
7. Hamhalter, J.: Spectral order of operators and range projections. J. Math. Anal. Appl. **331**, 1122–1134 (2007)
8. Hanche-Olsen, H., Stormer, E.: Jordan Operator Algebras. Pitman Advanced Publishing program. Pitman, Boston (1984)
9. Kadison, R.V.: Order properties of bounded self-adjoint operators. Proc. Am. Math. Soc. **2**, 505–510 (1951)
10. Kadison, R.V.: Isometries of operator algebras. Ann. Math. **54**, 325–338 (1951)
11. Kaplansky, I.: Any orthocomplemented complete modular lattice is a continuous geometry. Ann. Math. **61**(3), 524–541 (1955)
12. Olson, M.P.: The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice. Proc. Am. Math. Soc. **28**(2), 537–544 (1971)
13. Sherman, S.: Order in operator algebras. Am. J. Math. **73**, 227–232 (1951)